Fisher-Hartwig conjecture and the correlators in the inpenetrable Bose gas.

A.A.Ovchinnikov

Institute for Nuclear Research, RAS, Moscow, 117312, Russia

Abstract

We apply the theorems from the theory of Toeplitz determinants to calculate the asymptotics of various correlators including the exponential ones in the inpenetrable one-dimensional Bose gas system. The known correlators in the free-fermion system are also used to test the generalized Fisher-Hartwig conjecture.

1. Introduction

At present time the calculation of the correlators in the one-dimensional exactly solvable models including the constant for the asymptotic power-law behaviour remains an open problem. Except the correlators in the XY - spin chain and the Ising model the calculations are possible for the case of one-dimensional Bose gas with δ - function interaction [1] in the strong coupling limit [2]. In this case as well as in the case of the free-fermion system, the correlators are reduced to the Toeplitz determinants [2] which allows to calculate their asymptotics exactly.

In the present letter to calculate the asymptotics of the determinants for both systems we use the conjecture from the theory of Toeplitz matrices, [3], the generalized Fisher-Hartwig conjecture, which is based on the proofs of the original Fisher-Hartwig conjecture [4] including the constant in front of the asymptotics, in a number of the particular cases [5], [6] (see also [3] and references therein). One finds the representation of the generating function $f(x)$ of the Toeplitz matrix $M_{ij} = M(i-j) = \int_0^{2\pi}(dx/2\pi)e^{i(i-j)x}f(x)$ of the following form:

$$f(x) = f_0(x)\prod_r e^{ib_r(x-x_r-\pi\mathrm{sign}(x-x_r))}(2-2\cos(x-x_r))^{a_r}, \qquad (1)$$

where $x \in (0,2\pi)$ is implied, the discontinuities (jumps and zeroes or the power-law singularities) are at finite number of the points x_r, and $f_0(x)$ is the smooth non-vanishing function with the continuously defined argument at the interval $(0,2\pi)$. The function (1) is characterized by the parameters a_r, b_r at each point of discontinuity x_r. In general there are several representations of the form (1) for a given functions $f(x)$. To obtain the asymptotics of the determinant one should take the sum over the representations (1) corresponding to the minimal exponent $\sum_r(b_r^2 - a_r^2)$:

$$D(N) = \sum_{Repr.} e^{l_0 N} N^{\sum_r(a_r^2-b_r^2)} E \qquad (2)$$

where E is the constant independent of N,

$$E = \exp\left(\sum_{k=1}^{\infty} k l_k l_{-k}\right) \prod_r (f_+(x_r))^{-a_r+b_r} (f_-(x_r))^{-a_r-b_r} \tag{3}$$

$$\prod_{r\neq s}\left(1-e^{i(x_s-x_r)}\right)^{-(a_r+b_r)(a_s-b_s)} \prod_r \frac{G(1+a_r+b_r)G(1+a_r-b_r)}{G(1+2a_r)},$$

$l_k = \int_{-\pi}^{\pi}(dx/2\pi)e^{ikx}\ln(f_0(x))$, and the functions $f_\pm(x)$ are given by the equations

$$\ln f_+(x) = \sum_{k>0} l_{-k} e^{ikx}, \quad \ln f_-(x) = \sum_{k>0} l_k e^{-ikx},$$

where G is the Barnes G-function [7], $G(z+1) = G(z)\Gamma(z)$, $G(1) = 1$. In ref.[5] the theorem was proved for $a_r = 0$ and an arbitrary number of the discontinuities of the imaginary part of the magnitude less than 1/2, $|b_r| < 1/2$, however, there are many reasons to believe it to be true also in the case $|b_r| = 1/2$ [3]. For the case of an arbitrary single Fisher-Hartwig singularity the conjecture (2) was recently proved in ref.[6]. See ref.[3] for the complete list of the cases for which the rigorous proof of the conjecture (2) is available. Schematically the rigorous proofs in the particular cases [5] [6] go as follows. Suppose that equation (2) is fulfilled for some functions $f_1(x)$ and $f_2(x)$ of the class of piecewise continuous functions with continuously defined argument. Then the equation (2) is fulfilled for the function $f(x) = f_1(x)f_2(x)$. Thus it is sufficient to prove eq.(2) for the smooth function with the continuously defined argument, in which case it is reduces to the well known strong Szego theorem and for the singular function of the form (1) $f(x) = (1-z)^\alpha(1-1/z)^\beta$, $z = e^{ix}$, in which case the asymptotics is known exactly.

The goal of the present letter is two-fold. First, the calculation of the correlators in the one-dimensional Bose gas and the free-fermion system provide the additional tests of the generalized Fisher-Hartwig conjecture [3] in the cases when the rigorous proofs are not available. Second, we calculate the asymptotics of the two kinds of the exponential correlators which is interesting by itself since they can be compared with the predictions of various approaches to the calculation of the asymptotics of the correlation functions in the exactly solvable models such as the harmonic fluid approach [8] or the conformal field theory approach [9].

2. Calculation of the correlators.

The calculation of the correlators in the inpenetrable Bose gas or the free fermion systems are based on the representation of the Toeplitz determinant as a random matrix average over the unitary group. It is easy to prove the following equation [2]:

$$\det{}_M(M_{ij}) = \frac{1}{M!}\prod_{i=1}^{M}\int_0^{2\pi}\frac{dx_i}{2\pi}f(x_i)\prod_{i<j}|z_i - z_j|^2, \qquad z_i = e^{ix_i}, \tag{4}$$

where the function $f(x)$ corresponds to the matrix $M_{ij} = M(i-j)$. In fact, we have the following simple chain of equations:

$$\prod_{i=1}^{M}\int_0^{2\pi}\frac{dx_i}{2\pi}f(x_i)\prod_{i<j}|z_i-z_j|^2 = \prod_{i=1}^{M}\int_0^{2\pi}\frac{dx_i}{2\pi}f(x_i)\sum_{P,P'}(-1)^{P+P'}e^{i\sum_i x_i(Pi-P'i)} =$$

$$\sum_{P,P'}(-1)^{P+P'}\prod_i M(Pi-P'i) = M!\mathrm{det}\left(M_{ij}\right),$$

where $P, P' \in S_M$ are the permutations. Since the ground state wave functions for the free fermion system and the inpenetrable Bosons systems can be represented as

$$\psi_{ff}(x_1\ldots x_M) = \prod_{i<j}\sin(\pi x_{ij}/L), \qquad \psi_b(x_1\ldots x_M) = \prod_{i<j}|\sin(\pi x_{ij}/L)|, \tag{5}$$

where $x_{ij} = x_i - x_j$ and L is the length of the system, the correlators are represented as Toeplitz determinants according to the formula (4).

We begin with the the following equal-time correlation function defined in terms of the Bose creation and annihilation operators $\phi^+(x)$, $\phi(x)$:

$$G_\alpha(x) = \langle 0|\phi^+(x)e^{i\alpha N(x)}\phi(0)|0\rangle, \tag{6}$$

where α is an arbitrary parameter and $N(x)$ is the operator of the number of particles at the segment $(0, x)$. In the framework of the first-quantization this correlator has the following form:

$$G_\alpha(x) = M\prod_{i=2}^{M}\int_0^L dx_i e^{i\alpha\sum_{i=2}^{M}\theta(x-x_i)}\psi(x, x_2, ...x_M)\psi(0, x_2, ...x_M), \tag{7}$$

where the wave functions are normalized to unity and correspond to Bose- statistic. One can see that at $\alpha = 0$ the correlator (7) is the one-particle density matrix for the inpenetrable bosons system, while at $\alpha = \pi$ the correlator is the equal-time Green function (one-particle density matrix) of the free-fermion system.

In fact, in this case the integrand in eq.(7) reduces to the product of two free-fermion ground state wave functions (one should take into account the equation $\sin(\pi(y-x)/L) = \mathrm{sign}(y-x)(1/2)(2-2\cos(2\pi(y-x)/L))^{1/2}$, $0 < y < L$). First, we calculate the correlator (7) at $\alpha < \pi$. Clearly, taking into account the normalization of the wave functions (5), the equation $|\sin(\pi x/L)| = (1/2)(2-2\cos(2\pi x/L))^{1/2}$, and the equation

$$|\sin(\pi x_{ij}/L)|^2 = (1/4)|e^{i2\pi x_i/L} - e^{i2\pi x_i/L}|^2,$$

one finally obtains the determinant of the form (4):

$$G_\alpha(x) = \frac{1}{L}\mathrm{det}_{M-1}M_{ij}[f(y)]$$

where the function $f(y)$ $(0 < y < 2\pi)$ equals

$$f(y) = (e^{i\alpha}; 1)(y)(2-2\cos(y-2\pi x/L))^{1/2}(2-2\cos(y))^{1/2}, \tag{8}$$

where we denote $(e^{i\alpha};1)(y) = e^{i\alpha}\theta(y-x_r)+\theta(x_r-y)$, $x_r = 2\pi x/L$. This function should be represented in the form (1) which gives

$$f_0(y) = e^{ibx_r} = e^{i\alpha x/L}, \quad x_1 = 0, \quad a_1 = \frac{1}{2}, \quad b_1 = -b, \quad x_2 = x_r, \quad a_2 = \frac{1}{2}, \quad b_2 = b, \tag{9}$$

where $b = \alpha/2\pi < 1/2$. Substituting this function into the determinant (2) we finally obtain the following expression for the correlator:

$$G_\alpha(x) = e^{i\alpha(Mx/L-1/2)} G^2\left(\frac{3}{2}+b\right) G^2\left(\frac{3}{2}-b\right) \rho \frac{1}{(2M\sin(\pi x/L))^{1/2+2b^2}},$$

where $b = \alpha/2\pi < 1/2$ and the particle density $\rho = M/L$. Note that the correlator (6) is the periodic function of the parameter α since at $\alpha > \pi$ ($b > 1/2$) one should use the function of the form (1) with the parameters $b_r = \pm(1-b)$ in order to have the minimal exponent in the equation (2). From this equation at $b = 0$ we obtain the asymptotics of the one-particle density matrix for the inpenetrable Bose gas [2]:

$$G(x) = \rho \frac{1}{(2M\sin(\pi x/L))^{1/2}} G^4(3/2).$$

Now let us turn to the calculation of the free-fermion Green function which corresponds to the value $\alpha = \pi$ or $b = \alpha/2\pi = 1/2$. In contrast to the case $\alpha < \pi$ in this case there are two different representations of the function (8) in the form (1) which corresponds to the equal value of the exponent $\sum_r(a_r^2 - b_r^2)$ in the asymptotics (2). Namely, we have the same parameters as in eq.(9) except the two possible choises: $b_1 = -b_2 = 1/2$, $b_1 = -b_2 = -1/2$. Taking the sum of this two terms we finally obtain the correct expression

$$G_{ff}(x) = \frac{\sin(\pi Mx/L)}{L\sin(\pi x/L)},$$

which provides an important test of the generalized Fisher-Hartwig conjecture (3).

Now we calculate the exponential correlator $\langle 0|e^{i\alpha N(x)}|0\rangle$, where again the operator $N(x)$ is the operator of the number of particles at the segment $(0,x)$. In the first- quantization language this correlator takes the following form:

$$\langle 0|e^{i\alpha N(x)}|0\rangle = \prod_{i=1}^{M}\int_0^L dx_i e^{i\alpha\sum_{i=1}^{M}\theta(x-x_i)}|\psi(x_1,x_2,...x_M)|^2, \tag{10}$$

where the ground state wave function is normalized to unity. The correlators for the inpenetrable Bose gas system and the free-fermion system are equal to each other. First consider the case $\alpha < \pi$. As in the case of the one-particle density matrix, we represent the correlator as the Toeplitz determinant:

$$\langle 0|e^{i\alpha N(x)}|0\rangle = \det{}_M M_{ij}[f(y)], \tag{11}$$

where the function $f(y) = e^{i2\pi b}\theta(2\pi x/L - y) + \theta(y - 2\pi x/L)$, $b = \alpha/2\pi$, The representation of this function in the form (1) is

$$f_0(y) = e^{ibx_r} = e^{i\alpha x/L}, \quad x_1 = 0, \quad a_1 = 0, \quad b_1 = -b, \quad x_2 = x_r, \quad a_2 = 0, \quad b_2 = b,$$

where $x_r = 2\pi x/L$. Substituting this function into the equations (2), (3) we obtain the following result:

$$\langle 0|e^{i\alpha N(x)}|0\rangle = e^{i\alpha Mx/L}\frac{1}{(2M\sin(\pi x/L))^{2b^2}}G(1+b)G(1-b).$$

Next, consider the case $\alpha = \pi$. As for the correlator $G_\alpha(x)$, at $\alpha = \pi$ ($b = 1/2$) we have two different ways to represent the function $f(y)$ (11) in the form (1). Taking the sum of the two terms we obtain:

$$\langle 0|e^{i\pi N(x)}|0\rangle = \frac{2\cos(\pi Mx/L)}{(2M\sin(\pi x/L))^{1/2}}G(3/2)G(1/2),$$

which is in fact a real function of x.

Finally, we consider the asymptotics of the density-density correlation function $\Pi(x) = \langle 0|\rho(x)\rho(0)|0\rangle$, where $\rho(x)$ is the density operator. The expression of this correlator in terms of the ground state wave function reads:

$$\Pi(x) = M^2\prod_{i=3}^{M}\int_0^L dx_i|\psi(x, 0, x_3, ...x_M)|^2. \tag{12}$$

Clearly this correlator is the same for the inpenetrable Bose gas system and the free fermion system. Performing the calculations we obtain the following expression of (12) as a Toeplitz determinant:

$$\Pi(x) = \frac{4}{L^2}\sin^2(\pi x/L)\det{}_{M-2}M_{ij}[f(y)],$$

where the function $f(y)$ is represented in the canonical form (1) and corresponds to the values $a_1 = a_2 = 1$, $b_1 = b_2 = 0$. Calculating the determinant we obtain

$$\det{}_{M-2}M_{ij}[f(y)] = \frac{(M-2)^2}{4\sin^2(\pi x/L)}$$

and the leading order term in the asymptotics of the density-density correlation function in the thermodynamic limit is $\Pi(x) = \rho^2$, where the density $\rho = M/L$. Thus the generalized Fisher-Hartwig conjecture gives the correct result in this case too.

In conclusion, we have calculated both the exponential correlators and the usual ones in the one- dimensional Bose gas and the free-fermion systems. In the cases when the results can be obtained by the different methods, namely for the one-particle density matrices for the Bose and Fermi - systems, the predictions of the generalized Fisher-Hartwig conjecture [3] provide the additional tests of this conjecture. In the cases of various exponential correlators the predictions may be usefull in the context of the different approaches [8], [9] to the calculation of the asymptotics of the correlation functions.

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